COMBINING INDEPENDENT TESTS IN CASE OF TRIANGULAR AND CONDITIONAL SHIFTED EXPONENTIAL DISTRIBUTIONS

By

Abed El-Qader Salah Sulieman El-Masri

B. Sc. (Statistics)

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NOTATION

T_A denotes the statistic for the method with an abbreviation A. ϕ_A denotes the test based on T_A and p_i denotes the i-th p-value.

A	T_A
Sum of p_i's	$T_S = -\sum_{i=1}^{n} p_i / \sqrt{n}$
Fisher	$T_F = -2 \sum_{i=1}^{n} \ln\left(p_i\right) / \sqrt{n}$
Logistic	$T_L = -\sum_{i=1}^{n} \ln\left[p_i/(1-p_i)\right] / \sqrt{n}$
Inverse normal	$T_N = -\sum_{i=1}^{n} \Phi^{-1}(p_i) / \sqrt{n}$
Minimum of p_i's $1 \le i \le n$	$T_{min} = -\mathrm{Min}(p_i)$
Maximum of p_i's $1 \le i \le n$	$T_{max} = -\mathrm{Max}(p_i)$

Abbreviation

EBS : exact Bahadur slope

ABS : approximate Bahadur slope

LP : local power

iid : independent and identically distributed

w.p.1 : with probability 1

rv : random variable

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Criteria

* $C_A(\xi)$ denotes the exact Bahadur slope of the method with an abbreviation A at parameter value ξ

* $\varphi(x)$ and $\Phi(x)$ are the probability density function (pdf) and the distribution function (DF) of N(0,1) respectively

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ملخــــص

دمج الاختبارات المستقلة في حالتي التوزيع المثلثي والتوزيع الأسـي المنحــرف الشرطــي

اعــــداد

عبد القادر صلاح المصــــري

في هذه الرسالة سوف ندرس دمج n من الاختبارات المستقلة لفحص فرضية بسيطة عندما n تؤول الى ما لا نهاية في حالتي التوزيع المثلثي والتوزيع الأسي المنحرف الشرطي ، حيث سنقوم بدراسة الطرق المشهورة التالية :

أصغر قيم p ، اكبر قيم p ، الطبيعي المعكوس ، لوجستك ، فيشر ومجموع قيـم p وذلك من خلال مقارنتها بوساطة ميل بهادر التام . في حالة التوزيع المثلثـي ذا اقتران كثافة احتمالية

$$f(x)=(-b^2/2)\ x + b\ ,\quad 0 < x < 2/b\ ,\quad b \geqslant 2$$

عندما b تؤول الى 2 وجدنا أن مجموع قيم p افضل من جميع الطرق الاخـرى ، ثم الطبيعي المعكوس ، لوجيستك وفيشر بترتيب تنازلي وأن اسوأ طريقتين همـــا اصغر قيم p واكبر قيم p وايضا في حالة b تؤول الى ما لا نهاية وجدنـا ان مجموع قيم p افضل من جميع الطرق الاخرى ، ثم اكبر قيم p ، الطبيعــي المعكوس ، لوجيتك وفيشر بترتيب تنازلي . واسوأطريقة هي اصغر قيم p . وكذلـك قمنا بمقارنة ميل بهادر التام للاختبارات المختلفة في حالة ان b تنتمي الـى $[2 ، \infty)$.

في حالة التوزيع المنحرف الشرطي ذا اقتران كثافة احتمالية

$$f(x|\theta) = e^{-(x-\gamma\theta)}\ ,\ x \geq \gamma\theta\ ,\ \theta \in [a,\infty),\ a \geq 0.$$

وفي حالة ان $\theta_1, \theta_2, \ldots$ لهم حالتين توزيعيتين

الحالة (١) عندما $\theta_1, \theta_2, \ldots$ لهما اقتران كثافة متجمع عام ، حيث وجدنـا ان طريقة الطبيعي المعكوس افضل من جميع الطرق الاخرى في حالة ان المعلمه γ تؤول الى الصفر ، وبترتيب تنازلي لوجستك ، مجموع قيم p ، وفيشر وان اسـوأ طريقتين هما اصغر قيم p واكبر قيم p . وعندما كانت المعلمه تؤول الـى

ما لا نهاية وجدنا ان طريقة الطبيعي المعكوس افضل من جميع الطرق الاخرى ايضا، وبترتيب تنازلي لوجيستك ، اكبر قيم p وفيشر واسوأطريقة هي اصغر قيم p .

الحالة (٢) عندما $\theta_1, \theta_2, \cdots$ لها توزيع جاما G(1,2) ، وجدنا ان الطبيعي المعكوس افضل من جميع الطرق الاخرى عندما المعلمه γ تؤول الى ما لا نهاية وبترتيب تنازلي لوجيستك ، واكبر قيم p ، فيشر ومجموع قيم p ، واسوأطريقة هي اصغر قيم p .

وكذلك قمنا باجراء مقارنات عدديه في الحالتين

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ABSTRACT

In this thesis we will consider the problem of combining n independent tests as $n \longrightarrow \infty$ for testing a simple hypothesis in case of triangular and conditional shifted exponential distributions. We will study a number of popular combination methods viz., sum of p-values, inverse normal, logistic, Fisher, minimum of p-values and maximum of p-values. We will compare their performance via Exact Bahadur Slope.

In case of triangular distribution with pdf $f(x) = (-b^2/ 2) x + b$, $0 < x < (2/b)$, $b \geq 2$ we will find that as the parameter $b \longrightarrow 2$ the sum of p-values is better than all other methods, followed in decreasing order by the inverse normal, logistic and Fisher's method. The worst is the minimum and maximum of p-values methods. Also, as the parameter $b \longrightarrow \infty$ we will find that the sum of p-values is better than all other methods, followed in decreasing order by the maximum of p-values, the inverse normal, the logistic and the Fisher's methods. The worst is the method of minimum of p-values. Also comparisons between the EBS's of the tests

mentioned above have been made for b : $2 \leq b < \infty$.

In case of conditional shifted exponential with pdf

$$f(x|\theta) = e^{-(x-\gamma\theta)} \quad , \ x \geq \gamma\theta, \ \theta \in [a,\infty), \ a \geq 0$$

there are two cases to consider.

Case 1: When θ_1, θ_2, ... are distributed according to the distribution function DF, we will show that if the parameter $\gamma \longrightarrow 0$ then the inverse normal is better than the other methods and is followed in decreasing order by logistic, sum of p-values and Fisher's method. The worst is the minimum and maximum of p-values.

But if the parameter $\gamma \longrightarrow \infty$ then the inverse normal is better than all other methods, and is followed in decreasing order by logistic, maximum of p-values and Fisher's methos.

Case 2: When θ_1, θ_2, ... have the DF Gamma (1,2) we will show that as the parameter $\gamma \longrightarrow \infty$ the inverse normal is better than all other methods, and is followed in decreasing order by logistic, maximum of p-values, Fisher and sum of p-values. The worst is the minimum of p-values.

Also, we make some comparisons by numerical calculations in both cases.

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Abed El-Qader S. Al-Masri

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الى حضرة صاحب الجلالة الهاشمية

الحسين بن طلال

حفظه الله

اهديه عملي المتواضع هذا

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Contents

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CHAPTER 1

INTRODUCTION

1.1 PREFACE

The problem of combining independent tests of hypothesis is an important and also a popular statistical practice. There are many methods which are used for combining independent tests and they are compared by using different criteria viz., Exact Bahadur Slope (EBS), Approximate Bahadur Slope (ABS), Pitman Efficiency, Local Power, Admissibility and others. In this thesis we will study only six combination methods via EBS as the number of tests combined tends to infinity in two cases.

In the first case, we will consider the triangular distribution. In the second case, we will consider the conditional shifted exponential distribution.

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1.2 REVIEW OF THE LITERATURE

Several authors have considered the problem of combining n independent tests of hypothesis.

If H_0 is a simple hypothesis then Birnbaum (1955) showed that given any non-parametric combination method which has a monotone increasing acceptance region, there exists a problem for which this method is most powerful against some alternative.

Littell and Folks (1971), studied four methods of combining a finite number of independent tests. They found that the Fisher method is better than the inverse normal method, the minimum of p-values method and maximum of p-values via Bahadur efficiency.

Later, Littell and Folks (1973) studied all methods of combining a finite number of independent tests. They found that Fisher's method is optimal under some conditions.

Brown, Cohen and Strawderman (1976) have shown that such tests form a complete class.

Bataineh (1990) studied the problem of combining n independent tests as $n \longrightarrow \infty$. In case of shifted exponential distribution, he looked at a number of popular combination methods (inverse normal, logistic, Fisher, sum of p-values, minimum of p-values, and maximum of p-values) and compared their performance via EBS, ABS and LP. Also, he proved that the performance of no combination method is uniformly most powerful, via EBS and ABS; but via LP the combination methods, $C_S(\theta)$, $C_L(\theta)$ and $C_N(\theta)$ are equivalent.

Abu-Dayyeh and Bataineh (1992) showed that Fisher's

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method is strictly dominated by the sum of p-values method via EBS in case of combining infinite number of independent shifted expoenetial tests when the sample size of each tests remains finite.

Again Abu-Dayyeh (1992) showed that under contain conditions, that the local limit of exact Bahadur efficiency is equivalent to Pitman efficiency in case of shifted exponential distribution.

1.3 Specific Problems

Suppose that we have n simple hypotheses

$$H_0^{(i)}: \eta_i = \eta_0^i \quad \text{vs} \quad H_1^{(i)}: \eta_i \in \Omega_i - \{ \eta_0^i \} \qquad (1.3.1)$$

$i = 1, \ldots, n$ such that $H_0^{(i)}$ is rejected for large values of some continuous rv $T^{(i)}$, $i = 1, \ldots, n$. We want to combine the n hypotheses into one in the following way:

$$H_0: (\eta_1, \ldots, \eta_n) = (\eta_0^1, \ldots, \eta_0^n), \text{ vs}$$
$$H_1: (\eta_1, \ldots, \eta_n) \in \Omega_1 \times \ldots \times \Omega_n - \{(\eta_0^1, \ldots, \eta_0^n)\} \qquad (1.3.2)$$

There are many methods for combining several tests of hypothesis into one overall tests. Among these methods are the omnibus methods which correspond to combining the p-values of the different tests. The p-value of the i-th test is given by

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$$U_i(t) = P^i_{H_0}[\, T^{(i)} > t \,] = 1 - F^i(t),\ i = 1,\ \dots,\ n \qquad (1.3.3)$$

where $F^{(i)}$ is the DF of $T^{(i)}$ under $H_0^{(i)}$. Then note that under $H_0^{(i)}$ the rv $U_i \sim u(0,1)$, $i = 1, \dots, n$ and under $H_1^{(i)}$ the rv U_i has same distribution for $i = 1, \dots, n$ and this distribution is not $u(0,1)$.

In this thesis we will consider the special case: $\eta_i = \theta$ and $\eta_0^{(i)} = 0$ for $i = 1, \dots, n$ and $T^{(1)}, \dots, T^{(n)}$ are independent. Then (1.3.2) reduces to

$$H_0: \theta = 0 \text{ vs } H_1: \theta \in \Omega - \{0\} \qquad (1.3.4)$$

Also, the p-values $U_1, \dots, U_n$ are iid rv's which have a $u(0,1)$ distribution under H_0 and a distribution which is not $u(0,1)$ under H_1. Then the testing problem (1.3.4) is equivalent to

H_0: $U_1, \dots, U_n$ are iid $u(0,1)$, vs

H_1: $U_1, \dots, U_n$ are iid with pdf f which is not $u(0,1)$ but a support A which is a subset of $(0,1)$, (1.3.5)

In this thesis we will study the case where

$$f(u) = (-b^2/2)u + b,\ 0 < u < (2/b),\ b \ge 2$$

and we will study only six omnibus methods viz., maximum of p-values, minimum of p-values, Fisher, logistic, inverse normal and sum of p-values. Then (1.3.5) reduces to $U_1, \dots, U_n$ are iid rv's with pdf f and we want to test

H_0: $f \equiv u(0,1)$, vs H_1: $f \neq u(0,1)$ but the support of f is a subset of $u(0,1)$. (1.3.6)

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We will study the six methods via EBS when $n \longrightarrow \infty$ and this constitutes our first problem which is studied in chapter 2.

Next we will take the case

$$\eta_i = \gamma\, \theta_i \ , \quad i = 1, \ldots, n$$

where $\theta_1, \ldots, \theta_n$ are iid with DF F with support $[a,\infty)$, $a \geq 0$ and we want to test

$$H_0: \gamma = 0 \text{ vs } H_1: \gamma > 0 \qquad (1.3.7)$$

and where the i-th problem is based on $T_1^i, \ldots, T_{n_i}^i$ which are independent where pdf is given by $E(\gamma\theta_i,1)$. Then by sufficiency we can assume $n_i = 1$ for $i = 1, \ldots, n$. Thus the second problem that we will study in chapter 3 is: $T_1,\ldots,T_n$ are independent $E(\gamma\theta_i,1)$, and we want to test

$$H_0: \gamma = 0 \text{ vs } H_1: \gamma > 0 \qquad (1.3.8)$$

where $\theta_1,\ldots,\theta_n$ are iid with DF F with support $[a,\infty)$, $a \geq 0$. We want to study the same six methods (used in the first problem) via EBS as $n \longrightarrow \infty$. All the above methods viz.

$$-(\max p_i) \quad , \; -(\min p_i) \quad , \quad -2\sum_{i=1}^{n} \ln p_i \ ,$$

$$-\sum_{i=1}^{n} \ln\left(\frac{p_i}{1-p_i}\right) , \; -\sum_{i=1}^{n} \Phi^{-1}(p_i) \ , \; -\sum_{i=1}^{n} p_i$$

reject H_0 for large values of the test statistics.

1.4 Definitions and Preliminaries

In this section we will state some definitions and

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preliminaries that will be used later.

Definition 1.4.1. (Bahadur Efficiency and EBS)

Let $X_1, X_2, \ldots, X_n$ be iid from a distribution with pdf $f(x,\theta)$, and we want to test $H_0: \theta = \theta_0$, vs $H_1: \theta \in \Xi - \{\theta_0\}$. Let $\{T_n^1\},\{T_n^2\}$ be two sequences of test statistics for testing H_0. Let the significance level attained by T_n^i be $L_n^i = 1 - F_i(T_n^i)$ where $F_i(t_i) = P_{H_0}(T_n^i \le t_i)$, $i = 1, 2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^i\}$ such that $C_i(\theta) = \lim_{n\to\infty} -\frac{2}{n} \ln L_n^i$ w.p.1 under θ and the Bahadur efficiency of $\{T_n^1\}$ relative to $\{T_n^2\}$ is given by $\phi_{12} = C_1(\theta) / C_2(\theta)$. (For more details, see Serfling [9])

Theorem 1.4.1. (A large deviation theorem)

Let $X_1, X_2, \ldots, X_n$ be iid, with distribution F and put $S_n = \sum_{i=1}^{n} X_i$. Assume that the mgf (moment generating function) $M(z) = E_F\left(e^{zx}\right)$ exists in a neighbourhood of zero. Put $m(t) = \inf_z e^{-tz} M(z)$ then $\lim_{n\to\infty} - 2/n \ln p_F(S_n \ge nt) = - 2 \ln m(t)$. (See Serfling [9])

Theorem 1.4.2. (Bahadur Theorem)

Let $\left\{T_n\right\}$ be a sequence of test statistics which satisfies the following:

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1. Under H_1

$$\frac{T_n}{\sqrt{n}} \xrightarrow{a.s.} b(\theta) \quad \text{where } b(\theta) \in \mathbb{R}.$$

2. There exists an open interval I containing $\{b(\theta): \theta \in \Omega\}$, and a function g continuous on I such that

$$\lim_{n\to\infty} -\frac{2}{n} \ln\left[1 - F_n\left(\sqrt{n}\; t\right)\right] = g(t)$$

then the EBS is given by $C(\theta) = g(b(\theta))$.

(See Serfling [9])

The following theorems are from Abu Dayyah [1]. For more details see the reference.

Theorem 1.4.3.

Let $X_1, X_2, \ldots, X_n$ be iid with p.d.f $f(x,\theta)$ and we want to test $H_0: \theta = 0$, vs $H_1: \theta > 0$. For $j = 1, 2$ let $T_{n,j} = \sum_{i=1}^{n} f_j(x_i) / \sqrt{n}$ be a sequence of statistics such that H_0 will be rejected for large values of $T_{n,j}$ and let ϕ_j be the test based on $T_{n,j}$.

Assume $E_\theta(f_i(x)) > 0$, $\forall\, \theta \in \Theta$, $E_0(f_j(x)) = 0$, $var(f_i(x)) > 0$, the mgf $M_j(f_j(x))$ exists in a neighbourhood of zero and $f_j(x)$ is differentiable for $j = 1,2$ then

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1. If $b_j'(0)$ is finite for $j = 1, 2$ then

$$\lim_{\theta\to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{\underset{\theta=0}{\mathrm{var}}(f_2(x))}{\underset{\theta=0}{\mathrm{var}}(f_1(x))} \left[\frac{b_1'(0)}{b_2'(0)} \right]^2$$

where $b_j(\theta) = E_\theta(f_j(x))$, and $C_j(\theta)$ is EBS of test ϕ_j at θ.

2. If $b_j'(0)$ is infinite for some $j = 1, 2$ then

$$\lim_{\theta\to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{\underset{\theta=0}{\mathrm{var}}(f_2(x))}{\underset{\theta=0}{\mathrm{var}}(f_1(x))} \left[\lim_{\theta\to 0} \frac{b_1'(\theta)}{b_2'(\theta)} \right]^2$$

Theorem 1.4.4.

If $T_n^{(1)}$ and $T_n^{(2)}$ are two test statistics for testing H_0: $\theta = 0$, vs H_1: $\theta > 0$ with distribution functions $F_0^{(1)}$ and $F_0^{(2)}$ under H_0 respectively, and that $T_n^{(1)}$ is at least as powerful as $T_n^{(2)}$ at θ for any level α, then if ϕ_j is the test based on $T_n^{(j)}$, $j = 1,2$ then

$$C_{\phi_1}^{(1)}(\theta) \geq C_{\phi_2}^{(2)}(\theta)$$

Corollary 1.4.1. If T_n is the uniformly most powerful test $\forall$ α, then it is the best test via EBS.

Theorem 1.4.5.

$$2t \leq m_s(t) \leq et \quad , \qquad \forall t: 0 \leq t \leq 0.5$$

where

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$$m_s(t) = \inf_{z>0} \left\{ e^{-zt} \frac{1-e^{-z}}{z} \right\}$$

Theorem 1.4.6.

1. $m_L(t) \geq 2\, t\, e^{-t}, \quad \forall\, t \geq 0$

2. $m_L(t) \leq t\, e^{1-t}, \quad \forall\, t \geq 0.852$

3. $m_L(t) \leq t \left[t^2 / (1+t^2) \right]^3 e^{1-t}, \quad \forall\, t \geq 4$

   where

   $$m_L(t) = \inf_{z\in(0,1)} \left\{ e^{-tz}\, \pi\, z \quad CSC\,(\pi\, z) \right\}$$

and CSC is an abbreviation for cosecant function.

Theorem 1.4.7.

For $x > 0$,

$$\varphi(x) \left(\frac{1}{x} - \frac{1}{x^3} \right) \leq 1 - \Phi(x) \leq \varphi(x) / x$$

Theorem 1.4.8.

For $x > 0$

$$1 - \Phi(x) > \frac{\varphi(x)}{x + \sqrt{\pi/2}}$$

1.5 Summary of the Results

In this thesis, we will study the combination tests from the point of view of EBS for the triangular distribution and

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also for the conditional shifted exponential distribution.

The thesis is divided into three chapters. In chapter 1, we present the testing problems under consideration. Also we give a historical review of the related literature. Then we state some definitions and preliminaries that will be used in the thesis.

In Chapter 2, we will consider the problem (1.3.6), which is stated again below.

Suppose that the p-values $U_1, \ldots, U_n$ are iid rv's which have a u(0,1) distribution under H_0 and a distribution which is not u(0,1) under H_1.

i.e., H_0: $U_1, \ldots, U_n$ are iid u(0,1), vs

H_1: $U_1, \ldots, U_n$ are iid with p.d.f f which is not u(0,1) but has a support A which is a subset of (0,1).

In this chapter we will study the case where

$$f(u) = (-b^2/2)\ u + b,\ 0 < U < 2/b,\ b \geq 0.$$

We study the behaviour of the tests mentioned in the previous sections via EBS.

In particular we prove

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$\lim_{b\to 2} \dfrac{C_{max}(b)}{C_\phi(b)} = 0$ where $C_\phi(b)$ refer to any one of $C_S(b)$, $C_L(b)$, $C_N(b)$, $C_F(b)$, and $\dfrac{C_S(2)}{C_N(2)} = 1.084774707$, $\dfrac{C_N(2)}{C_L(2)} = 1.078030255$, $\dfrac{C_L(2)}{C_F(2)} = 1.5616979860$. Also, we will show that

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_\phi(b)} = 1 \text{ where } C_\phi(b) \in \left\{C_S(b), C_L(b), C_N(b), C_F(b)\right\}.$$

Further more we will show that

1. $C_s(b) > C_{max}(b)$, $\forall\ b \geq 2$
2. $C_{max}(b) > C_N(b)$, $\forall\ b \geq 6$
3. $C_N(b) > C_L(b)$ for large b
4. $C_L(b) > C_F(b)$ for large b

In Chapter 3, we will consider problem (1.3.8), which is stated again below.

Suppose that we test

$$H_0: \gamma = 0, \text{ vs } H_1: \gamma > 0$$

where the i-th problem is based on $T_1, \ldots, T_n$ which are independent rvs from conditional shifted exponential $E(\gamma\theta_i, 1)$ and where θ_1, θ_2, ... are iid with DF F with support $[a, \infty)$, $a \geq 0$. We study the behaviour of the tests mentioned in the previous sections via EBS. In section 3.3.2, we will consider $\theta_1, \theta_2, \ldots$ with general DF F with support $[a, \infty)$ and prove that

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i. $\lim_{\gamma \to 0} C_{max}(\gamma)/C_{\phi}(\gamma) = 0,$

where $C_{\phi}(\gamma) \in \left\{C_S(\gamma), C_N(\gamma), C_L(\gamma), C_F(\gamma)\right\}$ and

$$\lim_{\gamma \to 0} \frac{C_F(\gamma)}{C_S(\gamma)} = \frac{1}{3} \quad , \quad \lim_{\gamma \to 0} \frac{C_S(\gamma)}{C_L(\gamma)} = \lim_{\gamma \to 0} \frac{C_S(\gamma)}{C_N(\gamma)} =$$

$$\lim_{\gamma \to 0} \frac{C_F(\gamma)}{C_L(\gamma)} = \lim_{\gamma \to 0} \frac{C_F(\gamma)}{C_N(\gamma)} = \lim_{\gamma \to 0} \frac{C_L(\gamma)}{C_N(\gamma)} = 0$$

$$\lim_{\gamma \to \infty} \frac{C_L(\gamma)}{C_F(\gamma)} = 1 \quad , \quad \lim_{\gamma \to \infty} \frac{C_F(\gamma)}{C_N(\gamma)} = 0 \quad , \quad \lim_{\gamma \to \infty} \frac{C_L(\gamma)}{C_N(\gamma)} = 0$$

$$\lim_{\gamma \to \infty} \frac{C_{max}(\gamma)}{C_F(\gamma)} = \lim_{\gamma \to \infty} \frac{C_{max}(\gamma)}{C_L(\gamma)} = a / E_F \theta \text{ and } \lim_{\gamma \to \infty} \frac{C_{max}(\gamma)}{C_N(\gamma)} = 0$$

In section (3.3), we will prove that $\lim_{\gamma \to \infty} \frac{C_S(\gamma)}{C_{\phi}(\gamma)} = 0$ for $C_{\phi}(\gamma) \in \left\{C_N(\gamma), C_F(\gamma), C_L(\gamma)\right\}$. If $\theta_1, \theta_2, \ldots$ have a Gamma(1,2) or u(0,1) distribution and if $\theta_1, \theta_2, \ldots$ have $E(\theta,1)$ distribution then $\lim_{\gamma \to \infty} \frac{C_S(\gamma)}{C_F(\gamma)} = \frac{1}{2}$.

In the end we will make some numerical comparisons.

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CHAPTER 2

EXACT BAHADUR SLOPE FOR TRIANGULAR DISTRIBUTION

2.1 Introduction

In this chapter we will study the testing problem (1.3.6). We will compare the six methods viz., sum of the p-values, maximum of p-values, minimum of p-values, logistic, inverse normal and Fisher via EBS.

2.2 Derivation of the EBS

In this section, we will study the behaviour of the tests mentioned in chatper 1 via EBS in case of the following problem.

Suppose that the p-values $U_1, \ldots, U_n$ are iid rv's shich have a u(0,1) distribution under H_0 and a distribution which is not u(0,1) under H_1.
i.e., $H_0 = U_1, \ldots, U_n$ are iid with p.d.f f which is not u(0,1) but has a support A which is a subset of (0,1). In this chapter we will study the case where

$$f(u) = (-b^2/2)\, u + b, \; 0 < u < 2/b, \; b \geq 2.$$

The p-value in this case is given by

$$P_n = P_0(U_n \leq u_n) = u_n \; .$$

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The EBS's for the tests given in chapter 1 are reported in the following theorem.

Result (2.2.1):

A(1). $C_F(b) = 1 + 2\ln b - 2\ln(3 - 2\ln 2 + 2\ln b)$ (2.2.1)

A(2). $C_s(b) = -2\ln m_s(b_s(b))$, where $b_s(b) = -2/3b$

and $$m_s(b_s(b)) = \inf_{z\in(0,\infty)} \left\{ e^{-zb_s(b)} \frac{(1-e^{-z})}{z} \right\} . \quad (2.2.2)$$

A(3). $C_L(b) = -2\ln m_L(b_L(b))$, where

$b_L(b) = (b/2-1)^2 \ln(b-2) - b^2/4 \ln b + b\ln b - \ln 2 + b/2$

and

$$m_L(b_L(b)) = \inf_{z\in(0,1)} \left\{ e^{-zb_L(b)} \pi z \csc(\pi z) \right\} . \quad (2.2.3)$$

$$A(4).\ C_N(b) = \frac{b^4}{16\pi} \left[\Phi\left(\sqrt{2}\ \Phi^{-1}(2/b)\right) \right]^2 . \quad (2.2.4)$$

A(5). $C_{max}(b) = 2 \ln b - 2 \ln 2$. (2.2.5)

A(6). $C_{min}(b) = 0$. (2.2.6)

The first four statements can be proved in a similar manner. Therefore, we will prove A(2) only.

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Proof of A(2)

$$\frac{T_s}{\sqrt{n}} = -\sum_{i=1}^{n} \frac{U_i}{n} \xrightarrow{w.p.1} b_s(b) \text{ under } b \text{ where}$$

$$b_s(b) = E(\,-U\,) = -\int_0^{2/b} u\,(-b^2/2\ u + b)\ du = -\,2/(3b),\ b \ge 2$$

Thus theorem (1.4.1) and Theorem (1.4.2) gives

$$C_s(b) = -\,2\,\ln m_s(b_s(b))$$

$$= -\,2\,\ln\left\{\inf_{z>0} e^{2z/(3b)}(1 - e^{-z})\,/\,z\right\}.$$

For the proof of A(5) and A(6) we need the following theorem (see Abu-Dayyeh [1]).

Theorem 2.2.1.

Let U_1, U_2, ... be iid rv's. We want to test H_0: $U_i \sim u(0,1)$, vs, H_1: $U_i \sim f$ on $(0,1)$, which is not $u(0,1)$. Then

1. $C_{max}(f) = -\,2\,\ln(\text{ess. Sup}_f(u))$

   where $\text{ess.Sup}_f(u) = \text{Sup}\{u:\ f(u) > 0\}$ w.p.1 under f.

2. If $t(\ln t)^2\, f(t) \longrightarrow 0$ as $t \longrightarrow 0$, then $C_{min}(f) = 0$.

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Proof of A(5)

By the above theorem

$C_{max}(b) = -2\ \ln(\text{ess. Sup}_b(u))$

where $\text{ess.Sup}_b u = \text{Sup}\left\{u:\ f(u) > 0\right\}$ w.p.1 under b.

For $f(u) = -b^2/2\ u + b$, $0 < u < 2/b$,

$\text{ess.Sup}_b u = 2\ /\ b$. Therefore,

$$C_{max}(b) = -2\ \ln(2/b) = 2\ \ln\ b - 2\ \ln\ 2.$$

Proof of A(6)

$$\lim_{t\to 0}\ t\ (\ln\ t)^2\ f(t) = \lim_{t\to 0}\ t(\ln\ t)^2\ \left[-\ b^2/2\ t + b\right]$$

$$= b \lim_{t\to 0}\ t(\ln\ t)^2 - b^2/2 \lim_{t\to 0}\ t^2(\ln\ t)^2\ .$$

Clearly, by using L'Hopital rule twice, $\lim_{t\to 0}\ t(\ln\ t)^2 = 0$ which implies $C_{min}(b) = 0$.

Now we will find the limits of the ratios of every pair of these slopes as $b \longrightarrow 2$ and as $b \longrightarrow \infty$. This gives the following results:

$$\lim_{b\to 2} \frac{C_{max}(b)}{C_\phi(b)} = 0, \text{ where } C_\phi(b) \in \left\{C_S(b), C_N(b), C_L(b), C_F(b)\right\}.$$

and

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$$\frac{C_S(2)}{C_N(2)} = 1.084774707 \quad , \quad \frac{C_N(2)}{C_L(2)} = 1.078030255$$

and $$\frac{C_L(2)}{C_F(2)} = 1.5616979860 \quad \text{(see Table 1).}$$

Also, $$\lim_{b\to\infty} \frac{C_{max}(b)}{C_\phi(b)} = 1 \text{ where } C_\phi(b)\in\left\{C_S(b),C_N(b),C_L(b),C_F(b)\right\}$$

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_{min}(b)} = \infty .$$

Proof:

By using Theorem (1.4.6) (2):

$$m_L(b_L(b)) \le b_L(b)\, e^{1-b_L(b)}, \ \forall\, b_L(b) > 0.852 . \qquad (2.2.7)$$

Thus

$$C_L(b) \ge -\,2\,\ln\left\{b_L(b)\, e^{1-b_L(b)}\right\}, \ \forall\, b_L(b) > 0.852 \qquad (2.2.8)$$

which implies

$$\frac{C_{max}(b)}{C_L(b)} \le \frac{C_{max}(b)}{-2\ln b_L(b)-2+2b_L(b)} \le \frac{C_{max}(b)}{2b_L(b)}$$

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_L(b)} \le \lim_{b\to\infty} \frac{-2\ln 2+2\ln b}{2b_L(b)}$$

$$= \lim_{b\to\infty} \frac{-\ln 2+\ln b}{b_L(b)}$$

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$$= \lim_{b\to\infty} \frac{1}{b b_L'(b)} \text{ , since } \lim_{b\to\infty} b_L(b) = \infty$$

by using L'Hopital's rule.

But

$$b_L'(b) = 1 + \left(\frac{b-2}{2}\right) \ln\left(\frac{b-2}{b}\right) \text{ and therefore}$$

$$\lim_{b\to\infty} b b_L'(b) = \lim_{b\to\infty} \left[b + \frac{b}{2}(b-2) \ln\left(\frac{b-2}{b}\right)\right] = 1 . \qquad (2.2.9)$$

and hence

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_L(b)} \leq 1 . \qquad (2.2.10)$$

Again by Theorem (1.4.6)

$$C_L(b) \leq -2\ln\left(2\, b_L(b)\, e^{-b_L(b)}\right) , \qquad \forall\ b_L(b) \geq 0 \quad (2.2.11)$$

and therefore

$$\frac{C_{max}(b)}{C_L(b)} > \frac{-2\ln 2 + 2\ln b}{-2\ln 2 - 2\ln b_L(b) + 2b_L(b)} .$$

which implies that

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_L(b)} \geq \lim_{b\to\infty} \left[1 \Big/ \left(-\frac{b b_L'(b)}{b_L(b)} + b b_L'(b)\right)\right] = 1 . \quad (2.2.12)$$

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Hence from (2.2.10) and (2.2.12) we have

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_L(b)} = 1 .$$

Nwxt by (2.2.4) and (2.2.5) we have

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_N(b)} = \lim_{b\to\infty} \frac{2 \ln b - 2 \ln 2}{\frac{b^4}{16\pi}\left\{ \Phi\left(\sqrt{2}\ \Phi^{-1}\left(\frac{2}{b}\right)\right)\right\}^2} \equiv \lim_{b\to\infty} g(b) .$$

Let $y = -\Phi^{-1}(2/b) \longrightarrow b = 2 / \Phi(-y)$. Thus as $b \longrightarrow \infty$ we get $y \longrightarrow \infty$. (2.2.13)

Then

$$g(y) = \frac{-2\pi \ln(1-\Phi(y))\,(1-\Phi(y))^4}{\left[1 - \Phi\left(\sqrt{2}\ y\right)\right]^2} . \qquad (2.2.14)$$

By Theorem (1.4.8)

$$(1-\Phi(y))^4 \geq \left[\frac{\varphi(y)}{y+\sqrt{\pi/2}}\right]^4 \qquad (2.2.15)$$

Also, $1 - \Phi(y) \leq \dfrac{\varphi(y)}{y}$ (2.2.16)

which implies

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$$- 2 \ln(1-\Phi(y)) \geq y^2 + 2 \ln y + \ln(2\pi) \ . \qquad (2.4.17)$$

Finally,

$$\left(1-\Phi\left(\sqrt{2}\, y\right)\right)^2 \leq \Phi^2\left(\sqrt{2}\, y\right) / 2y^2 \qquad (2.2.18)$$

Thus $\quad g(y) \geq \dfrac{y^2(2\ln y + \ln 2\pi + y^2)}{\left(y + \sqrt{\pi/2} \right)^4}$

$$\lim_{y\to\infty} g(y) \geq 1 \qquad (2.2.19)$$

By (2.2.14), (2.2.15) and (2.2.16)

$$g(y) \leq \frac{\left(\ln 2\pi + y^2 + 2\ln\left(y+\sqrt{\pi/2}\right)\right) \left(\sqrt{2}\, y+\sqrt{\pi/2}\right)^2}{2\, y^4}$$

Then

$$\lim_{y\to\infty} g(y) \leq 1 \qquad (2.2.20)$$

Hence, from (2.2.19) and (2.2.20) we have

$$\lim_{b\to\infty} \frac{C_{max}(b)}{C_N(b)} = 1 \ .$$

Thus as $b \longrightarrow \infty$ all tests are equivalent.

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The proof of the remaining limits is similar to the proof given above.

Now we will compare the above EBS's for $2 < b < \infty$.

Result 2.2.2.

$$C_s(b) > C_{max}(b) \quad \text{for } \forall\ b \ge 2.$$

Proof: By Theorem (1.4.5) we have

$C_s(b) \ge -2\ \ln(e\ b_s(b)) = -2 - 2\ln b_s(b)$ and by (2.2.5)

Therefore,

$$\begin{aligned} C_s(b) - C_{max}(b) &\ge -2 - 2\ln(2/3b) + 2\ln 2 - 2\ \ln b \\ &= -2 - 2\ln 2 + 2\ \ln 3 + 2\ln b + 2\ \ln 2 - 2\ \ln b \\ &= 2\ \ln 3 - 2 > 0 \end{aligned}$$

$$\longrightarrow \quad C_s(b) \ge C_{max}(b)\ , \quad \forall\ b \ge 2.$$

Result 2.2.3.

$$C_{max}(b) > C_N(b) \qquad \forall\ b > 6.$$

Proof: By (2.2.4) and (2.2.5) we get

$$g(b) = C_{max}(b) - C_N(b) = 2\ln b - 2\ln 2 - \frac{b^4}{16\pi}\left\{\Phi\left[\sqrt{2}\ \Phi^{-1}(2/b)\right]\right\}^2$$

By (2.2.13), we get

$$g(y) = -2\ln(1 - \Phi(y)) - \frac{\left\{1 - \Phi\left[\sqrt{2}\ \ y\right]\right\}^2}{\pi\left\{1-\Phi(u)\right\}^4}$$

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By Theorem (1.4.7), (2.2.16), (2.2.17) and (2.2.18) we have

$g(y) = \ln 2\pi + 2 \ln y$, $\forall\, y > 0$, i.e., $\forall\, b > 4$.

Then $g(y) > 0$ if $\ln 2\pi + 2 \ln y > 0$

$$\text{if } b > 2 \Big/ \left[\Phi(-1/\sqrt{2\pi}\,) \right] \cong 6.$$

$$\longrightarrow \qquad C_{max}(b) > C_N(b) \quad , \qquad b \geq 6$$

Result 2.2.4.

$$\lim_{b \to \infty} \left[C_N(b) - C_L(b) \right] \geq 0 .$$

Proof: By (2.2.3), (2.2.4) and (2.2.11) we get

$$g(b) \equiv C_N(b) - C_L(b) \geq \frac{b^4}{16\pi} \left\{ \Phi\left[\sqrt{2}\, \Phi^{-1}(2/b)\right] \right\}^2 + 2\ln\, b_L(b) + 2 - 2b_L(b)$$

By (2.2.13), we get

$$g(y) \geq \frac{1}{\pi(1-\Phi(y))^4} \left\{ 1 - \Phi\left(\sqrt{2}\ \ y\right) \right\}^2 + 2\ln\, b_L\left(\frac{2}{1-\Phi(y)}\right) + 2 - 2b_L\left(\frac{2}{1-\Phi(y)}\right) \qquad (2.2.21)$$

Now

$$b_L\left(\frac{2}{1-\Phi(y)}\right) = \left[\frac{1}{1-\Phi(y)} - 1\right]^2 \ln\left[\Phi(y)\right] - \ln\left[1-\Phi(y)\right] + \frac{1}{1-\Phi(y)} .$$

Then by (2.2.15), (2.2.16), (2.2.17) and (2.2.18) we get

$$b_L\left(\frac{2}{1-\Phi(y)}\right) < \left[\frac{y+\sqrt{\pi/2}}{\varphi(y)}\right] + \left[\frac{y+\sqrt{\pi/2}}{\varphi(y)}\right]^2 \cdot \ln\left[1 - \frac{\varphi(y)}{y+\sqrt{\pi/2}}\right] + \frac{1}{2}\ln 2\pi + y^2/2 + \ln\left[1+\sqrt{\pi/2}\right] \qquad (2.2.22)$$

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we have

$$\frac{\left[1-\Phi\left(\sqrt{2}\,y\right)\right]^2}{\left[1-\Phi(y)\right]^4} > \frac{2\pi y^4}{\left[\sqrt{\pi/2} + \sqrt{2}\,y\right]^2} . \qquad (2.2.23)$$

By (2.2.21), (2.2.22) and (2.2.23) we get

$$\lim_{y\to\infty} g(y) = \infty$$

and

$\lim_{b\to\infty} \left[C_N(b) - C_L(b) \right] = \infty$. This completes the proof of result (2.2.4).

Result 2.2.5.

$$\lim_{b\to\infty} \left[C_L(b) - C_F(b) \right] \geq 0 .$$

Proof: By (2.2.8) and (2.2.1) we have

$$g(b) \equiv C_L(b) - C_F(b) \geq - 2 \ln b_L(b) - 2 + 2 b_L(b) - 1 - 2 \ln b + 2 \ln(3 - 2 \ln 2 + 2 \ln b) .$$

Now

$$2 \lim_{b\to\infty} (b_L(b) - \ln b) = (3 - 2 \ln 2),$$

$$\lim_{b\to\infty} 2 \ln\left[\frac{3-2\ln 2+2\ln b}{b_L(b)}\right] = 2 \ln \left[\lim_{b\to\infty} \frac{2}{b b_L'(b)} \right] ,$$

Then (2.2.9) and (2.2.10) imply

$$\lim_{b\to\infty} \left[C_L(b) - C_F(b) \right] \geq - 3 + (3 - 2 \ln 2) + 2 \ln 2 = 0$$

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Note that from theorem (2.2.5) $C_N(b)$ is greater than $C_L(b)$ for large b, and from theorem (2.2.6) $C_L(b)$ is greater than $C_F(b)$ for large b.

2.3 SUMMARY OF THE NUMERICAL RESULTS

Table (1) gives the performance of EBS's for different values of b: b ∈ [2,15]. We observe that when b = 2 the numerical calculations verify the mathematical results:

$$C_S(b) > C_N(b) > C_L(b) > C_F(b) > C_{max}(b) = C_{min}(b)$$

But for $b \longrightarrow \infty$ we could not verify the mathematical results numerically because of the difficulty to get the values of EBS's on the computer.

The behavior of EBS's in different intervals is given below:

b ∈ [2,2.25]: $C_S(b) > C_N(b) > C_L(b) > C_F(b) > C_{max}(b)$

b ∈ [2.5,2.75]: $C_S(b) > C_N(b) > C_L(b) > C_{max}(b) > C_F(b)$

b ∈ [3,15]: $C_S(b) > C_{max}(b) > C_N(b) > C_L(b) > C_F(b)$

Summary: For this problem of combining independent tests of hypothesis, we showed that the T_s combination is better than all other combination methods in case of triangular distribution via EBS.

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In this problem, by using the limits and numerical results, we have shown that the maximum of p-values is worst than all other combination methods at minimum value of the parameter b = 2.

Also, by difinition of efficiancy and by using numerical results, we find that sum of p-values is better than inverse normal method, also inverse normal is better than the logistic method and the logistic method is better than Fisher's method which implies that the sum of p-values is better than all other methods as $b \longrightarrow 2$.

But for different values of b, we can see in Table (1) that the sum of p-values is better than all others. Also, we provve that the sum of p-values is batter than the maximum of p-values for all values of b, and all other methods are worst than maximum of p-values, which implies that the sum of p-values is the best.

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CHAPTER 3

EXACT BAHADUR SLOPE FOR CONDITIONAL SHIFTED EXPONENTIAL DISTRIBUTION

3.1 INTRODUCTION

In this chapter we will study the following testing problem: suppose that we test

$$H_0: \gamma = 0, \text{ vs } H_1: \gamma > 0$$

where the i-th problem is based on $T_1, \ldots, T_n$ which are independent random variable from conditional shifted exponential with pdf $f(x|\theta) = e^{-(x-\gamma\theta)}$, $x \geq \gamma\theta$ and $\theta_1, \ldots, \theta_n$ are iid with DF F with support $[a,\infty)$, $a \geq 0$. Also, we will study the same tests that we studied in chapter 2 via EBS.

3.2 DERIVATION OF THE EBS WITH GENERAL D.F. F OF θ.

In this section we will study the behaviour of the tests mentioned in Chapter 1 via EBS.

The p-value in this case is given by

$$P_i = P(X \geq x_i) = e^{-x_i}.$$

For the tests given in Chapter 1, the EBS's are given in the following theorem.

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Result (3.2.1)

A(1). $C_F(\gamma) = 2\gamma E_F\theta - 2 \ln[1 + \gamma E_F\theta]$ (3.2.1)

A(2). $C_S(\gamma) = - 2 \ln m_S(1/2 E_F(e^{-\gamma\theta})),$

where $m_S(t) = \inf_{z>0} \left\{ e^{-tz} (1-e^{-z}) / z \right\}$ (3.2.2)

A(3). $C_L(\gamma) = - 2 \ln m_L(b_L(\gamma))$

where $m_L(\gamma) = \inf_{0<z<1} \left\{ e^{-b_L(\gamma)z} \pi z \, CSC(\pi z) \right\}$ (3.2.3)

and $b_L(\gamma) = \gamma E_F(\theta) - E_F(e^{\gamma\theta} - 1) \ln(1 - e^{-\gamma\theta})$

A(4). $C_N(\gamma) = \left[E_F\left\{e^{\gamma\theta} \varphi\left[\Phi^{-1}\left(e^{-\gamma\theta}\right)\right]\right\}\right]^2$ (3.2.4)

A(5). $C_{max}(\gamma) = 2\gamma a$ (3.2.5)

A(6). $C_{min}(\gamma) = 0$. (3.2.6)

Proof of A(3)

$$\frac{T_L}{\sqrt{n}} = - \frac{1}{n} \sum_{i=1}^{n} \ln\left[\frac{P_i}{1-P_i} \right]$$

$$\longrightarrow b_L(\gamma) \equiv - E\left[\ln\left(\frac{P_i}{1-P_i} \right) \right] = - EE\left[\ln\left(\frac{e^{-x}}{1-e^{-x}} \right) \middle| \theta\right]$$

$$= \gamma E_F(\theta) - E_F\left(e^{\gamma\theta}-1\right) \ln\left(1-e^{-\gamma\theta}\right) \text{ w.p.1 under } \gamma.$$

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(For more details see Bataineh [5]).

Also, by Theorem (1.4.1) and Theorem (1.4.2) we get

$$C_L(\gamma) = -2\ \ln\left\{ \inf_{0<z<1} \left[e^{-b_L(\gamma)z} \pi z \, CSC(\pi z) \right] \right\}.$$

Proof of A(5): By Theorem (2.2.2)

Suppose $g(\theta)$ is the pdf pof the DF F. Then the joint pdf of x and θ is

$h(x,\theta) = g(\theta)\ f(x|\theta)$ where $f(x|\theta) = e^{-(x-\gamma\theta)}$, $x > \gamma\theta$.

Then the marginal pdf of x is

$$f(x) = \int_a^{x|\gamma} h(x,\theta)\ d\theta = \int_a^{x|\gamma} e^{-(x-\gamma\theta)} g(\theta)\ d\theta\ ,\ x > a\gamma,\ a \geq 0$$

$$= \begin{cases} e^{-x} \displaystyle\int_a^{x|\gamma} e^{\gamma\theta}\ dF(\theta)\ , & x > \gamma\theta \\ 0 & \text{otherwise.} \end{cases}$$

Then under γ the p-value $= e^{-x} = P$

satisfies : $0 < P < e^{-\gamma a}$

$$\text{ess. sup } P = e^{-a\gamma}$$

which implies $C_{max}(\gamma) = 2\gamma a$ by theorem (2.2.1).

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Proof of A(6)

$$g(p) = \int_{a}^{-(\ln p)/\gamma} e^{\gamma\theta} g(\theta)\, d\theta$$

then $\lim_{p\to 0} p(\ln p)^2 g(p)$

$$= \lim_{p\to 0} \frac{(\ln p)^2}{1/p} \int_{a}^{-(\ln p)/\gamma} e^{\gamma\theta} g(\theta)\, d\theta$$

$$= \lim_{p\to 0} -p^2 \left[(2\ln p)/p \int_{a}^{-(\ln p)/\gamma} e^{\gamma\theta} g(\theta)\, d\theta + \frac{(\ln p)^2}{p} g\left(\frac{-\ln p}{\gamma}\right)\right].$$

$$= \lim_{p\to 0} -2p\ln p \int_{a}^{-(\ln p)/\gamma} e^{\gamma\theta} g(\theta)\, d\theta = 0$$

using L'hopital rule since $g(\infty) = 0$ and $\lim_{p\to 0} p(\ln p)^2 = 0$

Now we will start comparing the EBS's.

Firstly, we find the limits of the ratios of every pair of these slopes as $\gamma \longrightarrow 0$ and as $\gamma \longrightarrow \infty$. This gives the following results

A(1). $\lim_{\gamma\to 0} \frac{C_S(\gamma)}{C_L(\gamma)} = \lim_{\gamma\to 0} \frac{C_L(\gamma)}{C_N(\gamma)} = \lim_{\gamma\to 0} \frac{C_{max}(\gamma)}{C_F(\gamma)} = 0$

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A(2). $\lim_{\gamma \to 0} \frac{C_F(\gamma)}{C_S(\gamma)} = \frac{1}{3}$

B(1). $\lim_{\gamma \to \infty} \frac{C_L(\gamma)}{C_F(\gamma)} = 1$

B(2). $\lim_{\gamma \to \infty} \frac{C_F(\gamma)}{C_N(\gamma)} = \lim_{\gamma \to \infty} \frac{C_L(\gamma)}{C_N(\gamma)} = 0.$

The proofs of equalities in A(1) and in A(2) are similar. Therefore, we will prove one of them which is $\lim_{\gamma \to 0} \frac{C_S(\gamma)}{C_L(\gamma)} = 0$. Also the proofs of the equalities in B(1) and B(2) are similar and therefore we will prove $\lim_{\gamma \to \infty} \frac{C_L(\gamma)}{C_F(\gamma)} = 1$.

Proof

By theorem (3.2.1)

$$b_S(\gamma) = \frac{1}{2} E_F\left[e^{-\gamma\theta}\right], \quad (3.2.7)$$

$$b_S'(\gamma) = -\frac{1}{2} E_F\left[\theta\ e^{-\gamma\theta}\right] \quad (3.2.8)$$

and

$$b_L'(\gamma) = -E_F\left(\theta\ e^{\gamma\theta}\ \ln(1-e^{-\gamma\theta})\right) \quad (3.2.9)$$

These imply

$$b_S'(0) = -E_F\ \theta \text{ finite}, \quad b_L'(0) = \infty .$$

Thus by part (2) of theorem (1.4.3)

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$$\lim_{\gamma\to 0} \frac{C_S(\gamma)}{C_L(\gamma)} = \frac{Var_{\gamma=0}(\text{logistic})}{Var_{\gamma=0}(\text{sum of p-value})} \left[\lim_{\gamma\to 0} \frac{-E_F\theta/2}{E_F\theta e^{\gamma\theta} \ln(1-e^{-\gamma\theta})} \right]^2$$

$$= 0$$

Now we will prove $\displaystyle \lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} = 1$

By (2.2.12), (3.2.3) and (3.2.1)

$$\frac{C_L(\gamma)}{C_F(\gamma)} \le \frac{-2\ln 2 - 2\ln b_L(\gamma) + 2b_L(\gamma)}{2\gamma E_F(\theta) - 2\ln(1+\gamma E_F\theta)}$$

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \le \lim_{\gamma\to\infty} \frac{-2\ln 2 - 2\ln b_L(\gamma) + 2b_L(\gamma)}{2\gamma E_F(\theta) - 2\ln(1+\gamma E_F\theta)}$$

$$= \lim_{\gamma\to\infty} \frac{-2b_L'(\gamma)/b_L(\gamma) + 2b_L'(\gamma)}{2E_F(\theta) - 2\dfrac{E_F\theta}{1+\gamma E_F\theta}} \text{, by L'Hopital rule}$$

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \le 1 \qquad (3.2.10)$$

where

$$\lim_{\gamma\to\infty} b_L(\gamma) = \infty \quad , \quad \lim_{\gamma\to\infty} b_L'(\gamma) = E_F\ \theta \qquad (3.2.11)$$

Similarly using (2.2.8), (3.2.3) and (3.2.1), we can show that

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \ge 1 \ . \qquad (3.2.12)$$

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Hence from (3.2.10) and (3.2.12) we get $\lim_{\gamma \to \infty} \frac{C_L(\gamma)}{C_F(\gamma)} = 1.$

Result 3.2.2.

$$\lim_{\gamma \to \infty} \left[C_L(\gamma) - C_F(\gamma) \right] \geq 0.$$

Proof: By (3.2.1) and (2.2.8)

$$C_F(\gamma) - C_L(\gamma) \leq 2\gamma E_F(\theta) - 2\ln(1+\gamma E_F\theta) + 2\ln(b_L(\gamma)) + 2 - 2b_L(\gamma)$$

$$= 2\gamma E_F(\theta) - 2b_L(\gamma) + 2 + 2\ln \left(\frac{b_L(\gamma)}{1+\gamma E_F\theta} \right)$$

$$\lim_{\gamma \to \infty} \frac{b_L(\gamma)}{1+\gamma E_F(\theta)} = \lim_{\gamma \to \infty} \frac{b_L'(\gamma)}{E_F(\theta)} = 1 \text{ by (3.2.11)}$$

$$\lim_{\gamma \to \infty} 2 \left(\gamma E_F\theta - b_L(\gamma) \right) = 2 (-1) = -2 \text{ , by L'Hopital rule}$$

Then $\lim_{\gamma \to \infty} \left(C_F(\gamma) - C_L(\gamma) \right) \leq 0$

$\longrightarrow C_F(\gamma) \leq C_L(\gamma)$ for large γ.

From the above relations we conclude that locally as $\gamma \to 0$

$$C_N(\gamma) > C_L(\gamma) > C_S(\gamma) > C_F(\gamma) > C_{max}(\gamma) > C_{min}(\gamma)$$

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But as $\gamma \longrightarrow \infty$, we conclude that only

$$C_N(\gamma) > C_L(\gamma) > C_{max}(\gamma) > C_F(\gamma) > C_{min}(\gamma)$$

As for as $C_S(\gamma)$ is concerned, we can't conclude any thing for general prior F because $\lim\limits_{\gamma \to \infty} b_S'(\gamma) / b_S(\gamma)$ has an indeterminate form (0/0). Therefore, we will consider certain priors, namely, U(0,1), $G(\alpha,\beta)$ and $E(\theta,1)$.

3.3 THE EBS'S WITH SPECIFIC D.F. F OF θ

Result 3.3.1

$$\lim_{\gamma \to \infty} \frac{C_S(\gamma)}{C_F(\gamma)} = \lim_{\gamma \to \infty} \frac{-\, b_S'(\gamma) | b_S(\gamma)}{E_F\theta} \qquad (3.3.1)$$

Proof: By (3.2.7), (3.2.8) and Theorem (1.4.5)

$$\lim_{\gamma \to \infty} \frac{C_S(\gamma)}{C_F(\gamma)} \leq \lim_{\gamma \to \infty} \frac{-2\ln 2 - 2\ln b_S(\gamma)}{2\gamma E_F\theta - 2\ln(1+\gamma E_F\theta)}$$

$$= \lim_{\gamma \to \infty} \frac{-2b_S'(\gamma) | b_S(\gamma)}{2E_F\theta - 2\dfrac{E_F\theta}{1+\gamma E_F\theta}} = \lim_{\gamma \to \infty} \frac{-\, b_S'(\gamma) | b_S(\gamma)}{E_F\theta}$$

Similarly from (3.2.7), (3.2.8) and theorem (1.4.5)

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$$\lim_{\gamma\to\infty} \frac{C_S(\gamma)}{C_F(\gamma)} \geq \lim_{\gamma\to\infty} \frac{-b_S'(\gamma)|b_S(\gamma)}{E_F\theta}$$ which proves theorem (3.3.1). Now we will take the special priors. From (3.3.1) we conclude that

1. $\theta \sim U(0,1)$: $\lim_{\gamma\to\infty} \frac{-b_S'(\gamma)|b_S(\gamma)}{E_F\theta} = 0.$ (3.3.2)

2. $\theta \sim G(\alpha,\beta)$: $\lim_{\gamma\to\infty} \frac{-b_S'(\gamma)|b_S(\gamma)}{E_F\theta} = 0.$ (3.3.3)

3. $\theta \sim E(\theta,1)$: $\lim_{\gamma\to\infty} \frac{-b_S'(\gamma)|b_S(\gamma)}{E_F\theta} = 0.5$ (3.3.4)

As a special case for pdf of θ, let $\theta \sim G(\alpha,\beta)$ with $\alpha = 1$, $\beta = 2$. Here we want to determine the performance of $C_S(\gamma)$ with respect to another EBS's.

From result (3.3.1), we conclude that $C_S(\gamma) < C_F(\gamma)$ as $\gamma \longrightarrow \infty$.

By result (3.2.1) the EBS's of the tests under study when $\theta \sim G(1,2)$ are as follows:

$$C_F(\gamma) = 4\gamma - 2\ \ln(1+2\gamma) \qquad (3.3.5)$$

$$C_S(\gamma) = -2\ln\ m_S\left[\frac{1}{2(1+2\gamma)}\right]$$

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$$m_S(t) = \inf_{z>0} \left\{ e^{zt} \left[\frac{(1-e^{-z})}{z} \right] \right\} \quad (3.3.6)$$

$$C_L(\gamma) = -2 \ln m_L(b_L(\gamma))$$

,

$$b_L(\gamma) = 2\gamma - \frac{1}{2} \int_0^\infty \left(e^{\gamma\theta}-1\right) \ln\left(1-e^{-\gamma\theta}\right) e^{-\theta/2} \, d\theta$$

$$m_L(t) = \inf_{0<z<1} \left\{ e^{-tz} \, 2\pi \, CSC(\pi z) \right\} \quad (3.3.7)$$

$$C_N(\gamma) = \left[\frac{1}{2} \int_0^\infty e^{\theta(\gamma-1/2)} \, \varphi\left[\Phi^{-1}\left[e^{-\gamma\theta}\right]\right] d\theta \right]^2 \quad (3.3.8)$$

$$C_{max}(\gamma) = 0 \quad (3.3.9)$$

Finally, we make numerical calculations for the EBS of these procedures for different values of γ (see Table 2).

3.4 Summary of the Numerical Results

Table (2) gives the performance of EBS's for different values of γ: $\gamma \in [0.05, 20]$. We observe that when $\gamma = 0.05$ the numerical claculations verify the mathematical results

$$C_N(\gamma) > C_L(\gamma) > C_S(\gamma) > C_F(\gamma) > C_{max}(\gamma) = C_{min}(\gamma)$$

But for $\gamma \longrightarrow \infty$ we could not verify the mathematical results numerically because of the difficulty to get the values of EBS's on computer.

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The behavior of the EBS's in different intervals is given below:

$\gamma = 0.05$: $C_N(\gamma) > C_L(\gamma) > C_S(\gamma) > C_F(\gamma)$

$\gamma \in [0.1, 0.5]$: $C_S(\gamma) > C_N(\gamma) > C_L(\gamma) > C_F(\gamma)$

$\gamma = 1$: $C_N(\gamma) > C_L(\gamma) > C_F(\gamma) > C_S(\gamma)$

$\gamma \in [2,3]$: $C_L(\gamma) > C_F(\gamma) > C_N(\gamma) > C_S(\gamma)$

$\gamma \in [5,8]$: $C_F(\gamma) > C_L(\gamma) > C_N(\gamma) > C_S(\gamma)$

$\gamma \in [10,20]$: $C_N(\gamma) > C_F(\gamma) > C_L(\gamma) > C_S(\gamma)$

Summary: We showed that in case of shifted exponential distribution via EBS the T_n combination is better than all other combination methods.

We use only the limits of different result Bahadur slopes, as the parameter $\gamma \longrightarrow 0$ or as $\gamma \longrightarrow \infty$ and two cases for distribuion of θ_1, θ_2, ... In all the above cases we find that the inverse normal method is better than all other methods. But for other values of the parameter γ we use the numerical calculations to show that the inverse normal is the best in this case.

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Appendix

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TABLE (1)

THE EXACT BAHADUR SLOPES FOR TRIANGULAR DISTRIBUTION

	$C_s(b)$	$C_L(b)$	$C_F(b)$	$C_{max}(b)$	$C_N(b)$
000	.345294513517	.295269900523	.189069783783	000000000000	.31830988620
001	.346010935042	.295842737991	.189403144842	.000999750084	.31881753914
005	.348877746187	.298123267007	.190584234571	.004938522518	.32135531530
010	.352463664099	.300952547311	.192405842819	.009975083021	.32450830010
030	.366831397420	.312099546434	.199093398683	.029777224986	.33705339701
050	.381232085407	.323041059496	.205799576725	.049385225180	.34930664992
100	.417332797411	.349730765095	.222632154525	.095803283374	.37971636620
150	.453987834649	.375680056865	.239536010362	.144641323161	.40987578431
200	.489648272035	.401023207905	.256483984756	.190620359609	.43975054813
250	.525712987138	.425835228244	.273452640334	.235566071313	.46803002580
500	.703077516369	.543277692697	.357986564978	.446287102629	.60832200250
750	.872927797011	.651563412199	.440934378859	.636907462237	.73721096040
000	1.03373091049	.752232750819	.521477956183	.810930216216	.83281740531
500	1.32786349829	.934719668418	.674192666479	1.11923157587	1.0835269370
.000	1.58872467673	1.09683458942	.815619199156	1.38629436112	1.2727269350
.000	2.03093024952	1.37541679568	1.06811424600	1.83258146375	1.5612802380
.000	2.39469723492	1.60957424578	1.18726944251	2.19722457734	1.8963135220
.000	2.70280567181	1.81177585641	1.48031569078	2.20552593699	2.1462351960
).00	3.41610101401	2.29319211890	1.94999187616	3.21887582467	2.4604714510
5.00	4.22703061876	2.86113017723	2.51578217167	4.02980604109	3.3678048851

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Table (2)

THE EXACT BAHADUR SLOPES FOR CONDITIONAL SHIFTED EXPONENTIAL

γ	$C_S(\gamma)$	$C_L(\gamma)$	$C_F(\gamma)$	$C_N(\gamma)$
0.050	.024854878579	.029840070161	.009379640391	.032009921342
0.100	.090301841654	.081784329275	.035356886423	.084473593752
0.200	.251222559124	.216770534233	.127055526758	.232291668263
0.300	.441441513382	.379599248586	.259992741509	.409577433915
0.400	.632874874139	.563366193559	.424426670195	.605687850679
0.500	.817277077652	.764426605277	.613705638828	.815260345902
1.000	1.58872445840	1.95979065491	1.80277542266	1.98299777937
2.000	2.60526097873	4.90017266949	4.78112417514	4.59608066963
3.000	3.27811632207	8.18648033649	8.10817970184	7.37182334056
5.000	4.18208490725	15.1631904413	15.2042094544	12.6266868817
8.000	5.05272104922	26.0620602769	26.3335733118	21.1931656324
10.00	5.47533923656	33.4908814141	33.9109551246	40.3568000374
20.00	6.81343849453	71.6401290838	72.5728558666	162.328375064

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Program 1.

Pascal program to find the values of $C_S(b)$

```
PROGRAM ABED;
VAR
{                    ***  VARIABLES  ***                          }
     ZZ,Z,WW,HH,KK,EE,BB,MM,M,NN,N,II,I,DD,CC,C,G,B   : REAL;
     JJ,BF,CF,CMAXB,DI,YY,CS                          : REAL;
     J                                               : INTEGER;
BEGIN
{                    * READ VALES OF b *                          }
     FOR J:=1 TO 20 DO
     BEGIN
     WRITE ('INPUT THE VALUE OF B:');
{                 * THIS LOOP FOR MINIMIZE SUM OF P-VALUE *}
     READLN(B);
{     BB:=(1-EXP(-B))/(2*B);}
{BB:=2.3444    }
               Z:=1.0E-2;
{               BB:=(1/(2*B))-EXP(-B)/(2*B);}
                M:= EXP(Z*B)*(1-EXP(-Z))/Z;
                I:=Z;
     WHILE Z<= 20 DO
     BEGIN
         Z:=Z+1.E-2;
         N:=EXP(Z*B)*(1-EXP(-Z))/Z;
         IF N < M THEN
         BEGIN
              M:=N;
              I:= Z;
{              JJ:=-1/(2*(1+2*B));}
         END;
     END;
     C:=-2*LN(M);
{                   * WRITE VALUE OF bs(b) *}
     WRITELN('Cs--------->', C);
{*****************************************************************}
{  ********** THIS LOOP FOR MINIMIZE LOGISTIC EBS *********}
```

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```
{ BB:= B*B*LN(B-2)/4-B*B*LN(B)/4+LN(B-2)-B*LN(B-2)+
           B*LN(B)-LN(2)+B/2;
           ZZ:=1.E-3;
           MM:=(EXP(-ZZ*BB)*ZZ*PI)/SIN(ZZ*PI);
           II:=ZZ;
  WHILE ZZ< 1.0000 DO
  BEGIN
          NN := (EXP(-ZZ*BB)*ZZ*PI)/SIN(ZZ*PI);
          IF NN < MM THEN
          BEGIN
              MM:= NN;
              II := ZZ;
          END;
          ZZ := ZZ+1.E-3
     END;
     CC:= -2*LN(MM);}
{ ***  WRITE MINIMUM VALUE OF LOGISTIC EBS   ***                   }
{    WRITELN('CL ------------------> ',CC);}
{-----------------------------------------------------------------}
{                 *****  WRITE VALUE OF Cf(b)  *******             }
{  WRITELN('--------->=',-2*LN(BB)+2*BB);
  WRITELN('BF--------->   ',2+4*B);
  WW:=-1-3*LN(2)-LN(PI);
     YY:=2*LN(3+LN(2*PI)+B*B+2*LN(B));
     EE:=WW+KK/HH+YY;
  WRITELN('G(t)------>',EE);
  WRITELN('L.T.------>',-2*LN(BB)-2+2*BB-1-2*LN(B)+2*LN(3-
               2*LN(2)+2*LN(B)));
  WRITELN('---------->',-2*DD/BB);
     CF:=1+2*LN(B)-2*LN(3-2*LN(2)+2*LN(B));
  WRITELN('CF---------->',-2*LN(1+2*B)+4*B);
  WRITELN('BL-------->',BB);
  WRITELN('MM--->',MM);}
{  JJ:=1/(2*B)-EXP(-B)/(2*B);
  WRITELN('JJ------>',JJ);}
                              END;
                                  END.
```

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Program number 2

```
{ *** THIS PROGRAM IS TO CALCULATE THE INTEGRATION VALUE OF
    bL FOR THE SECOND PROBLEM UNDER GAMMA DISTRIBUTION (1,2),
                                      BY USING SIMPSON'S RULE  ***}
PROGRAM ABED(NPUT,OUTPUT);
VAR
   RR,ZZ                                 : REAL;
   BL                                    : REAL;
{ ****************** FUNCTION FAY ********************  }
FUNCTION  FAY(VAR R:REAL;VAR Z : REAL):REAL;
FUNCTION  FNP(R,X: REAL) : REAL;
VAR AM,MN,BN:REAL;
BEGIN
  {****      DIVIDED THE FUNCTION INTO SEVERAL PARTS  ***** }
     AM:= EXP(R*X)-1;
     MN:=1-EXP(-R*X);
     BN:=EXP(-X/2.0);
     FNP := 0.5*(AM*LN(MN)*BN);
END;
VAR
{         ****   SIMPSON'S RULE      ***          }
   L0,U0,D1,P0,W0,W1,W2  :  REAL;
   N0,I : INTEGER;
BEGIN
   N0 := 1000;
   L0:=0.0009;
   U0 := Z;
   D1 :=(U0-L0)/N0;
   P0:=0;
   W0:=FNP(R,L0);
   FOR I:=1 TO N0 DO
   BEGIN
      W1:=FNP(R,L0+0.5*D1);
      L0:=L0+D1;
      W2:=FNP(R,L0);
      P0:=P0+D1*(W0+4*W1+W2)/6.0;
      W0:=W2;
```

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```
   END;
   FAY:=P0;
END;
BEGIN
     WRITELN;
     WRITE('INPUT THE VALUE OF R AND THE MAX Z');
     READLN(RR,ZZ);
     BL:=2*RR-FAY(RR,ZZ);
     WRITELN(BL);READLN;
END.
```

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Vita

Abed El-Qader Salah Sulieman El-Masri was born in AL-Noiema, Jordan in 1968. After receiving the General Secondary Certificate in 1986 he entered Yarmouk University where he earned a Bachelor Degree in Statistics in 1990.

In February 1990 he joined the Master Program of the Department of Statistics at Yarmouk University.

As a graduate student, he was also working as a teaching assistant in the Department of Statistics.

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